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A NOTE ON THE ASYMPTOTIC BEHAVIOR OF NONLINEAR SEMIGROUPS AND T--ETC(U)
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A NOTE ON THE ASYMPTOTIC BEHAVIOR OF NONLINEAR SEMIGROUPS AND THE RANGE OF ACCRETIVE OPERATORS

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A NOTE ON THE ASYMPTOTIC BEHAVIOR OF NONLINEAR SEMIGROUPS AND THE RANGE OF ACCRETIVE OPERATORS

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ABSTRACT

Let E be a Banach space, $A \subset E \times E$ an accretive operator that satisfies the range condition, J_r the resolvent of A, and S the nonlinear semigroup generated by $-A$. It is shown that if the norm of E^* is Fréchet differentiable, then $\lim_{t\to\infty} J_t x/t = \lim_{t\to\infty} S(t)x/t$ for each x in $c\ell(D(A))$, and $\lim_{t\to 0+} (x - J_t x)/t = \lim_{t\to 0+} (x - S(t)x)/t$ for each $x \in D(A)$. All limits are taken in the norm topology. If E is also smooth, then the first common limit is $-v$, where v is the unique point of least norm in $c\ell(R(A))$. If, in addition, A is closed, then the second common limit is A^0x, the unique point of least norm in Ax. We also show that if A is m-accretive and E^* is strictly convex, then $c\ell(R(A))$ is convex.

AMS (MOS) Subject Classification - 47H06, 47H09, 47H20

Key Words - Accretive operator, asymptotic behavior, nonexpansive mapping, resolvent, nonlinear semigroup.

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SIGNIFICANCE AND EXPLANATION

It is known that certain problems in partial differential equations may be interpreted as initial value problems for ordinary differential equations in Banach spaces. When such an evolution equation is governed by an accretive operator, then its solutions give rise to a nonlinear contraction semigroup. In this paper we study certain aspects of the asymptotic behavior of nonlinear semigroups and of resolvents of accretive operators. We also derive new results on their behavior at the origin. It turns out that the behavior of a nonlinear semigroup resembles that of the resolvent of its generator both at infinity and at the origin.

A NOTE ON THE ASYMPTOTIC BEHAVIOR OF NONLINEAR SEMIGROUPS AND THE RANGE OF ACCRETIVE OPERATORS

Simeon Reich*

1. INTRODUCTION.

Throughout this paper E denotes a (real) Banach space, $A \subset E \times E$ an accretive operator that satisfies the range condition, J_r the resolvent of A, and S the nonlinear semigroup generated by $-A$. The main purpose of the present paper is to sharpen some of the results of our previous paper [22] concerning the weak and strong convergence of $J_t x/t$ and $S(t)x/t$ as $t \to \infty$, and the properties of the range of A. The first results in this direction were established by Crandall (see [2, p. 166]) and Pazy [10] in Hilbert space. For recent developments in Banach spaces see the papers by Kohlberg and Neyman [8, 9] and the author [17, 18, 19]. In addition, we also derive new results on the weak and strong convergence of $(x - J_t x)/t$ and $(x - S(t)x)/t$ as $t \to 0+$.

In particular, we show that if the norm of E^* is Frechet differentiable, then the strong $\lim_{t\to\infty} J_t x/t = \lim_{t\to\infty} S(t)x/t$ for each $x \in c\ell(D(A))$, and the strong $\lim_{t\to 0+} (x - J_t x)/t = \lim_{t\to 0+} (x - S(t)x)/t$ for each $x \in D(A)$. If E is also smooth, then the first common limit is $-v$, where v is the unique point of least norm in $c\ell(R(A))$. If, in addition, A is closed, then the second common limit is A^0x, the unique point of least norm in Ax. We also show that if A is m-accretive and E^* is strictly convex, then $c\ell(R(A))$ is convex.

The asymptotic behavior of resolvents and nonlinear semigroups is studied in Sections 2 and 3. Nonexpansive mappings are treated in Section 4. Section 5 is devoted to the properties of $c\ell(R(A))$, and Section 6 to the behavior of $J_t x$ and $S(t)x$ at the origin.

Let E be a real Banach space, and let I denote the identity operator. Recall that a subset A of $E \times E$ with domain $D(A)$ and range $R(A)$ is said to be accretive if

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$|x_1 - x_2| \le |x_1 - x_2 + r(y_1 - y_2)|$ for all $[x_i, y_i] \in A$, $i = 1,2$ and $r > 0$. The resolvent $J_r : R(I + rA) \to D(A)$ and the Yosida approximation $A_r : R(I + rA) \to R(A)$ are defined by $J_r = (I + rA)^{-1}$ and $A_r = (I - J_r)/r$. We denote the closure of a subset D of E by $c\ell(D)$, its closed convex hull by $c\ell co(D)$, and its distance from a point x in E by $d(x,D)$. We also define $\|D\| = d(0,D)$. We shall say that A satisfies the range condition if $R(I + rA) \supset c\ell(D(A))$ for all $r > 0$. In this case, $-A$ generates a nonexpansive nonlinear semigroup $S : [0,\infty) \times c\ell(D(A)) \to c\ell(D(A))$ by the exponential formula: $S(t)x = \lim_{n\to\infty} \left(I + \frac{t}{n} A\right)^{-n} x$.

Recall that the norm of E is said to be Gateaux differentiable (and E is said to be smooth) if $\lim_{t\to 0} (|x + ty| - |x|)/t$ exists for each x and y in $U = \{x \in E : |x| = 1\}$. It is said to be uniformly Gateaux differentiable if for each y in U, this limit is approached uniformly as x varies over U. The norm is said to be Fréchet differentiable if for each x in U this limit is attained uniformly for y in U. We shall write that E is (UG) and (F) respectively. We shall need the following two known lemmata (cf. [6] and [15]).

Lemma 1.1. E^* is (F) if and only if E is reflexive and strictly convex, and has the following property: if the weak $\lim_{n\to\infty} x_n = x$ and $|x_n| \to |x|$, then $\{x_n\}$ converges strongly to x.

Lemma 1.2. E^* is (F) if and only if for any convex set $K \subset E$, every sequence $\{x_n\}$ in K such that $|x_n|$ tends to $d(0,K)$ converges.

Note that if E^* is smooth (strictly convex), then E is strictly convex (smooth). Hence, if E is reflexive, E is strictly convex (smooth) if and only if E^* is smooth (strictly convex). However this duality is not valid for all Banach spaces.

The duality map from E into the family of nonempty subsets of its dual E^* is defined by

$$J(x) = \{x^* \in E^* : (x,x^*) = |x|^2 = |x^*|^2\}$$

It is single-valued if and only if E is smooth. An operator $A \subset E \times E$ is accretive if and only if for each $x_i \in D(A)$ and each $y_i \in Ax_i$, $i = 1,2$, there exists

$j \in J(x_1 - x_2)$ such that $(y_1 - y_2, j) \geq 0$. We shall repeatedly use the following simple lemma.

Lemma 1.3. $(b - a, j) \geq |b|(|b| - |a|)$ for all $a, b \in E$ and $j \in J(b)$.

Proof. We know that $(|x + ty| - |x|)/t \geq (y, j)$ for all $t > 0$ and $j \in Jx/|x|$. To obtain the desired result, let $t = 1$, $x = b$, and $y = a - b$.

2. RESOLVENTS.

In this section we study the weak and strong convergence of $J_t x/t$ as $t \to \infty$.

Let x belong to $cl(D(A))$, and let $t > s > 0$. Since $(x - J_s x)/s$ belongs to $AJ_s x$ and $(x - J_t x)/t$ belongs to $AJ_t x$, we have

$$((x - J_s x)/s - (x - J_t x)/t, j) \geq 0$$

for some j in $J(J_s x - J_t x)$. Therefore

$$\left(\frac{1}{s}\left(x - J_s x - (x - J_t x)\right) + \left(\frac{1}{s} - \frac{1}{t}\right)(x - J_t x), j\right) \geq 0$$

and

$$\frac{1}{s}|J_s x - J_t x|^2 \leq |J_s x - J_t x| \frac{(t - s)}{st} |x - J_t x| .$$

It follows that

$$|J_s x - J_t x| \leq \left(1 - \frac{s}{t}\right)|x - J_t x| . \tag{2.1}$$

Applying Lemma 1.3 with $a = J_s x - J_t x$ and $b = x - J_t x$, we obtain $(x - J_s x, j) \geq \left(\frac{s}{t}\right)|x - J_t x|^2$ for all j in $J(x - J_t x)$. Hence

$$((x - J_s x)/s, j_t) \geq |x - J_t x|^2/t^2 \tag{2.2}$$

for all $j_t \in J((x - J_t x)/t)$. Let $d = d(0, R(A))$. Since $(x - J_t x)/t$ belongs to $R(A)$, $|x - J_t x|/t \geq d$. Consequently,

$$((x - J_s x)/s, j_t) \geq d^2 \tag{2.3}$$

for all $j_t \in J((x - J_t x)/t)$.

Let a subnet of $\{j_t\}$ converge weak-star as $t \to \infty$ to $j \in E^*$ (which depends on x and the subnet). We obtain

$$((x - J_s x)/s, j) \geq d^2 . \tag{2.4}$$

Since $\lim_{t \to \infty} |J_t x/t| = d$ [22, Lemma 2.1], it follows that $|j| = d$.

Now let $z_s = (x - J_s x)/s$, and let $\hat{z}_s$ be the natural image of z_s in E^{**}. Suppose that a subnet of $\{\hat{z}_s\}$ converges weak-star as $s \to \infty$ to $z^{**} \in E^{**}$. Clearly $|z^{**}| \leq d$. Since $(z^{**}, j) \geq d^2$, we also have $|z^{**}| \geq d$. Thus $|z^{**}| = d$ and $(z^{**}, j) = d^2$. In other words, z^{**} always belongs to $J_{E^*}(j)$. This fact leads to the following results.

Proposition 2.1. Let E be a Banach space, $A \subset E \times E$ an accretive operator that satisfies the range condition, J_r its resolvent, x a point in $c\ell(D(A))$, and w_t^{**} the natural image of $j_t x/t$ in E^{**}. If E^* is smooth, then the weak-star $\lim_{t\to\infty} w_t^{**}$ exists.

Theorem 2.2. Let E be a Banach space, $A \subset E \times E$ an accretive operator that satisfies the range condition, J_r the resolvent of A, and $d = d(0, R(A))$.

(a) If E is reflexive and strictly conves, then the weak $\lim_{t\to\infty} J_t x/t$ exists for each x in $c\ell(D(A))$ (and its norm equals d).

(b) If E^* is (F), then the strong $\lim_{t\to\infty} J_t x/t$ exists.

Proof. Part (a) follows from Proposition 2.1. Part (b) follows from Lemma 1.1 and Part (a) becuase $\lim_{t\to\infty} |J_t x/t| = d$.

We have already shown in [22] that the weak $\lim_{t\to\infty} J_t x/t$ *exists if* E is (UG), reflexive and strictly convex. We have also shown there that the strong $\lim_{t\to\infty} J_t x/t$ exists if E is (UG) and E^* is (F), or if E is uniformly convex. Since E is uniformly convex if and only if the norm of E^* is uniformly Fréchet differentiable, we see that Theorem 2.2 unifies and improves upon these results.

Now let $j_t \in J((x - J_t x)/t)$ and $k_t \in J((y - J_t y)/t)$. Suppose that a subnet of $\{j_t\}$ converges weak-star to j_1 and that a subnet of $\{k_t\}$ converges weak-star to j_2. Let $z_s = (x - J_s x)/s$ and let a subnet of $\{\hat{z}_s\}$ converge weak-star to $z^{**} \in E^{**}$. Then the corresponding subnet of the natural image of $(y - J_s y)/s$ also converges to z^{**}. We have $|z^{**}| = |j_1| = |j_2| = d$ and $(z^{**}, j_1) = (z^{**}, j_2) = d^2$. Therefore $d^2 = (z^{**}, (j_1 + j_2)/2) \leq |z^{**}||(j_1 + j_2)/2| = d|(j_1 + j_2)/2|$ and $|(j_1 + j_2)/2| = d$. This observation implies the following proposition.

Proposition 2.3. Let E be a Banach space, $A \subset E \times E$ an accretive operator that satisfies the range condition, and J_r the resolvent of A. If E^* is strictly convex,

then the weak-star $\lim_{t\to\infty} J((x - J_t x)/t)$ exists and is independent of $x \in cl(D(A))$.

It is now clear that if E is reflexive and smooth, then the weak $\lim_{t\to\infty} J((x - J_t x)/t$ exists (and has norm d). It follows that if E if reflexive and (F), then the strong $\lim_{t\to\infty} J((x - J_t x)/t$ exists.

3. NONLINEAR SEMIGROUPS.

In this section we study the weak and strong convergence of $S(t)x/t$ as $t \to \infty$.

Let x belong to $D(A)$, and let $t > s > 0$. It is known [11] that

$$|S(s)x - J_t x| \le \left(1 - \frac{s}{t}\right)|x - J_t x| + \left(\frac{2}{t}\right)\int_0^s |x - S(r)x|dr , \tag{3.1}$$

and that $|x - S(r)x| \le rp(x)$, where

$$p(x) = \lim_{r\to 0+} |x - J_r x|/r = \lim_{r\to 0+} |x - S(r)x|/r \le \|Ax\| .$$

Consequently,

$$|S(s)x - J_t x| \le \left(1 - \frac{s}{t}\right)|x - J_t x| + \left(\frac{s^2}{t}\right)p(x) . \tag{3.2}$$

Applying Lemma 1.3 with $a = S(s)x - J_t x$ and $b = x - J_t x$, we obtain

$$(x - S(s)x, j) \ge |x - J_t x|\left(\left(\frac{s}{t}\right)|x - J_t x| - \left(\frac{s^2}{t}\right)p(x)\right)$$

for all j in $J(x - J_t x)$. Hence

$$((x - S(s)x)/s, j_t) \ge |x - J_t x|^2/t^2 - \left(\frac{s}{t^2}\right)|x - J_t x|p(x) \tag{3.3}$$

for all $j_t \in J((x - J_t x)/t)$.

Denote $d(O,R(A))$ by d, and let a subnet of $\{j_t\}$ converge weak-star as $t \to \infty$ to $j \in E^*$. We obtain

$$((x - S(s)x)/s, j) \ge d^2 . \tag{3.4}$$

Since $|j| = d$, (3.4) implies that $|x - S(s)x|/s \ge d$ for all positive s. But we always have $\lim_{t\to\infty} |x - S(t)x|/t \le d$. This yields the following new result. It is valid in all Banach spaces.

Proposition 3.1. Let E be an arbitrary Banach space, $A \subset E \times E$ an accretive operator that satisfies the range condition, S the semigroup generated by $-A$, and $x \in cl(D(A))$. Then $\lim_{t\to\infty} |S(t)x/t| = d(0,R(A))$.

Now let $y_s = (x - S(s)x)/s$, and let $\hat{y}_s$ be the natural image of y_s in E^{**}. Let a subnet of $\hat{y}_s$ converge weak-star as $s \to \infty$ to $y^{**} \in E^{**}$. Then $|y^{**}| = d$ and $(y^{**},j) = d^2$. In other words, y^{**} belongs to $J_{E^*}(j)$. As in Section 2, this fact leads to the following results.

Proposition 3.2. Let E be a Banach space, $A \subset E \times E$ an accretive operator that satisfies the range condition, S the semigroup generated by $-A$, x a point in $cl(D(A))$, and v_t^{**} the natural image of $S(t)x/t$ in E^{**}. If E^* is smooth, then the weak-star $\lim_{t\to\infty} v_t^{**}$ exists.

Theorem 3.3. Let E be a Banach space, $A \subset E \times E$ an accretive operator that satisfies the range condition, S the semigroup generated by $-A$, and $d = d(0,R(A))$.

(a) If E is reflexive and strictly convex, then the weak $\lim_{t\to\infty} S(t)x/t$ exists for each x in $cl(D(A))$ (and its norm equals d).

(b) If E^* is (F), then the strong $\lim_{t\to\infty} S(t)x/t$ exists.

Proof. Part (a) follows from Proposition 3.2. Part (b) follows from Lemma 1.1 and Part (a) because $\lim_{t\to\infty} |S(t)x|/t = d$ by Proposition 3.1.

Theorem 3.3 improves upon [22, Theorem 3.3] and the remark following it. Its proof shows that $\lim_{t\to\infty} S(t)x/t = \lim_{t\to\infty} J_t x/t$. Note that the example in [9] and (4.1) show that the conditions imposed on E in Theorem 3.3 cannot be further weakened.

4. NONEXPANSIVE MAPPINGS.

Let C be a closed subset of a Banach space E and $T : C \to C$ a nonexpansive ($|Tx - Ty| \le |x - y|$ for all x and y in C) mapping. Assume that the accretive operator $A = I - T$ saatisfies the range condition and let J_r denote its resolvent. In this section we study the weak and strong convergence of $\{T^n x/n\}$ as $n \to \infty$.

Let S be the semigroup generated by $I - T$. Since it is known [21, p. 82] that

$$|S(n)x - T^n x| \leq \sqrt{n}\, |x - Tx| \tag{4.1}$$

for all n, the following results are immediate consequences of Proposition 3.2 and Theorem 3.3.

<u>Proposition 4.1</u>. Let C be a closed subset of a Banach space E and $T : C \to C$ a nonexpansive mapping. Assume that $I - T$ satisfies the range condition. Let x belong to C, and let u_n^{**} be the natural image of $T^n x/n$ in E^{**}. If E^* is smooth, then the weak-star $\lim_{n\to\infty} u_n^{**}$ exists.

<u>Theorem 4.2</u>. Let C be a closed subset of a Banach space E and $T : C \to C$ a nonexpansive mapping. Assume that $A = I - T$ satisfies the range condition and let $d = d(0,R(A))$.

(a) If E is reflexive and strictly convex, then the weak $\lim_{n\to\infty} T^n x/n$ exists for each x in C (and its norm equals d).

(b) If E^* is (F), then the strong $\lim_{n\to\infty} T^n x/n$ exists.

Theorem 4.2 is essentially due to Kohlberg and Neyman [9] who use a different argument. They also show that if E is not reflexive and strictly convex (or if E^* is not (F)), then there exists a nonexpansive mapping $T : E \to E$ such that $\{T^n x/n\}$ does not converge weakly (or strongly). The limits obtained in Proposition 4.1 and Theorem 4.2 equal those of Proposition 2.1 and Theorem 2.2.

A direct proof of Proposition 4.1 and Theorem 4.2 is also possible. Indeed, let x and y belong to C, and denote $d(0,R(A))$ by d. By definition, $y = J_t y + t(J_t y - TJ_t y)$, and

$$(1 + t)J_t y = y + tTJ_t y . \tag{4.2}$$

Therefore

$$\begin{aligned}
|T^{k+1}x - J_t y| &= \left(1 + \tfrac{1}{t}\right)|T^{k+1}x - J_t y| - \tfrac{1}{t}\,|T^{k+1}x - J_t y| \\
&\leq \left(1 + \tfrac{1}{t}\right)|T^{k+1}x - J_t y| + |T^{k+1}x|/t - |J_t y|/t \\
&= |T^{k+1}x + T^{k+1}x/t - y/t - TJ_t y| + |T^{k+1}x|/t - |J_t y|/t \\
&\leq |T^k x - J_t y| + 2|T^{k+1}x|/t + |y|/t - |J_t y|/t .
\end{aligned}$$

Summing from $k = 0$ to $k = n - 1$, we obtain

$$|x - J_t y| - |T^n x - J_t y| \geq n|J_t y|/t - n|y|/t - \left(\frac{2}{t}\right) \sum_{k=0}^{n-1} |T^{k+1}x| . \tag{4.3}$$

By Lemma 1.3, this implies that

$$(x - T^n x, j_t) \geq$$

$$|(x - J_t y)/t| \left(n|J_t y|/t - n|y|/t - \left(\frac{2}{t}\right) \sum_{k=0}^{n-1} |T^{k+1}x|\right)$$

for all $j_t \in J((x - J_t y)/t)$.

Dividing by n, and letting a subnet of $\{j_t\}$ converge weak-star as $t \to \infty$ to $j \in E^*$, we obtain

$$((x - T^n x)/n, j) \geq d^2 . \tag{4.4}$$

Since $\lim_{t\to\infty} |J_t v|/t = \lim_{n\to\infty} |T^n x|/n = d$ [22, Lemma 2.1 and Proposition 4.3], it follows that $|j| = d$. It is now clear how Proposition 4.1 and Theorem 4.2 can be deduced from (4.4). We also see that if E^* is strictly convex, then the weak-star $\lim_{t\to\infty} J((x - J_t y)/t)$ exists and is independent of x and y in C.

Combining (4.1), Theorem 4.2, and the proof of [15, Theorem 2.2], we see that Theorem 4.2 implies Theorem 3.3 when $A = I - T$. It also implies Theorem 3.3 for general A if we assume, in addition, either that $c\ell(D(A))$ is convex, or that E is smooth. Indeed, assume first that $c\ell(D(A))$ is convex. Define $T : c\ell(D(A)) \to c\ell(D(A))$ by $Ty = S(1)y$ for each y in $c\ell(D(A))$. Since $c\ell(D(A))$ is convex, $I - T$ satisfies the range condition and the appropriate $\lim_{n\to\infty} S(n)x/n$ exists by Theorem 4.2. The result is now seen to follow, once again, from the proof of [15, Theorem 2.2]. We remark in passing that if A is m-accretive and E^* is (F), then $c\ell(D(A))$ is indeed convex [16, p. 382]. Assume now that E is smooth and let J_r denote the resolvent of A. Let j denote the weak $\lim_{t\to\infty} J((x - J_t x)/t)$, which exists by Proposition 2.3. Since j is independent of x, we see that (2.4) holds for all x in $c\ell(D(A))$. Therefore

$$((J_{s/n}^{i}x - J_{s/n}^{i+1}x)/(\frac{s}{n}), j) \geq d^2$$

for all $0 \leq i \leq n - 1$. Summing these inequalities from $i = 0$ to $i = n - 1$, we obtain

$$((x - J_{s/n}^{n}x)/(\frac{s}{n}), j) \geq nd^2 .$$

This leads to (3.4) and to Theorem 3.3.

Finally, we remark in passing that the special case $A = I - T$ of Theorem 2.2 implies the theorem itself if $c\ell(D(A))$ is convex. To see this, let $T = J_1$ and note that for $t > 1$

$$J_t x = tJ_{t-1}^{I-T}x/(t - 1) - x/(t - 1) . \tag{4.5}$$

This relationship also shows that Corollary 1 of [20] is in fact equivalent to (a variant of) Theorem 1 there.

5. THE MINIMUM PROPERTY.

A closed subset D of a Banach space E is said to have the minimum property [10] if $d(0,c\ell co(D)) = d(0,D)$. Let $A \subset E \times E$ be an accretive operator that satisfies the range condition. In this section we show that if E^* is strictly convex, then $c\ell(R(A))$ has the minimum property. This provides another positive answer to a question of Pazy [10, p. 239]. Several applications are also included.

Assume that E^* is strictly convex, and let j be the weak-star $\lim_{t\to\infty} J((x - J_t x)/t)$. This limit exists by Proposition 2.3 and is independent of $x \in c\ell(D(A))$. If $A = I - T$ and T is nonexpansive, then the case $n = 1$ of (4.4) shows that

$$(x - Tx, j) \geq d^2 \tag{5.1}$$

for all x in C, where $d = d(0,R(A))$. In other words, $(z,j) \geq d^2$ for all $z \in R(A)$. Consequently, $(w,j) \geq d^2$ for all w in $c\ell co(R(A))$. Hence $|w|d = |w||j| \geq (w,j) \geq d^2$ and $c\ell(R(I - T))$ is seen to possess the minimum property. In order to extend this result to all accretive operators, let x and y belong to $c\ell(D(A))$, and let $t > s > 0$. Since

$$(\frac{1}{s}(J_t y - J_s x) + (\frac{1}{s} - \frac{1}{t})(x - J_t y) + \frac{1}{t}(x - y), j) \geq 0$$

for some j in $J(J_s x - J_t y)$, it follows that

$$|J_s x - J_t y| \leq (1 - \frac{s}{t})|x - J_t y| + (\frac{s}{t})|x - y| . \qquad (5.2)$$

((3.2) can be similarly extended.) Applying Lemma 1.3 with $a = J_s x - J_t y$ and $b = x - J_t y$, we obtain $(x - J_s x, j) \geq |x - J_t y|(\frac{s}{t})(|x - J_t y| - |x - y|)$ for all j in $J(x - J_t y)$. Hence

$$((x - J_s x)/s, j_t) \geq |x - J_t y|^2/t^2 - |x - J_t y||x - y|/t^2 \qquad (5.3)$$

for all $j_t \in J((x - J_t y)/t)$. Suppose that a subnet of $\{j_t\}$ converges weak-star to j and that a subnet of the natural image of $(x - J_s x)/s$ in E^{**} converges weak-star to z^{**}. Then $((x - J_s x)/s, j) \geq d^2$, $|j| = d$, $|z^{**}| = d$, and $(z^{**}, j) = d^2$. The discussion preceding Proposition 2.3 now shows that if E^* is strictly convex, then the weak-star $\lim_{t\to\infty} J((x - J_t y)/t)$ exists and is independent of x and y. Combining this fact with the proof of [22, Theorem 2.3] we obtain the following result.

Theorem 5.1. Let E be a Banach space, and let $A \subset E \times E$ be an accretive operator that satisfies the range condition. If E^* is strictly convex, then $c\ell(R(A))$ has the minimum property.

Recall that an accretive operator $A \subset E \times E$ is called m-accretive if $R(I + A) = E$. (It then follows that $R(I + rA) = E$ for all positive r.) For m-accretive A, Theorem 5.1 can be strengthened.

Theorem 5.2. Let E be a Banach space, and let $A \subset E \times E$ be m-accretive. If E^* is strictly convex, then $c\ell(R(A))$ is convex.

Proof. Combine Theorem 5.1 with the proof of [22, Theorem 2.7]. An alternative proof can be based on the fact that if A is m-accretive and A_r is its Yosida approximation, then $R(A) = R(A_r)$ for all $r > 0$.

It may be of interest to determine if the strict convexity of E^* is necessary for Theorems 5.1 and 5.2 to hold. (We have already seen in [22] that neither theorem is true in all Banach spaces.) Previous results in the direction of Theorem 5.2 were obtained by Rockafellar [24], Browder [3], and the author [13, 14].

When E^* is stricly convex, Theorem 5.1 identifies the limit in Theorems 2.2, 3.3 and 4.2 as $-v$, where v is the point of least norm in $c\ell(R(A))$. It also shows that the assumption that E is (UG) can be replaced by the weaker assumption that E is smooth in Theorems 3.4 and 3.6 of [22], and in Theorem 3 of [7]. These theorems deal with iterations of nonexpansive mappings, infinite products of resolvents, and a certain nonlinear evolution equation. We mention in particular the following results.

<u>Corollary 5.3</u>. *Let C be a closed convex subset of a Banach space E, $T : C \to C$ a nonexpansive mapping, S the semigroup generated by $-a(I - T)$, $a > 0$, and x a point in C. Assume that E is smooth and that E^* is (F), and let v be the point of least norm in $c\ell(R(I - T))$. Then*

(a) $\lim_{t\to\infty} dS(t)x/dt = -v$,

and

(b) if T is strongly nonexpansive, then $\lim_{n\to\infty} (T^n x - T^{n+1}x) = v$.

<u>Proof</u>. Since C is convex, $I - T$ satisfies the range condition and $c\ell(R(I - T))$ has the minimum property by Theorem 5.1. We know that $\lim_{t\to\infty} |dS(t)x/dt| = |v|$ and $\lim_{n\to\infty} |T^n x - T^{n+1}x| = |v|$ (see [1, Theorem 4.3] and [4, Proposition 1.2]). Therefore the result follows by Lemma 1.2.

Finally, let E be a Banach space, and let $A \subset E \times E$ be m-accretive. Assume that E is reflexive, smooth, and strictly convex. Since $c\ell(R(A))$ is convex by Theorem 5.2, the nearest point map $P : E \to c\ell(R(A))$ exists. Part (a) of Theorem 2.2 can now be used to show that $I - P$ is nonexpansive. This improves upon [22, Proposition 4.2].

<u>Remark</u>. Bruce Calvert and the author have recently shown that if a Banach space is not smooth, then there is an accretive $A \subset E \times E$ that satisfies the range condition such that $c\ell(R(A))$ does not possess the minimum property. Consequently, a reflexive Banach space E is smooth if and only if $c\ell(R(A))$ has the minimum property for all accretive $A \subset E \times E$ that satisfy the range condition.

6. BEHAVIOR AT THE ORIGIN.

In this section we use the ideas of Sections 2 and 3 to study the behavior of $J_t x$ and $S(t)x$ as $t \to 0+$.

Theorem 6.1. Let E be a Banach space, $A \subset E \times E$ an accretive operator that satisfies the range condition, J_r the resolvent of A, and S the semigroup generated by $-A$.

(a) If E is reflexive and strictly convex, then the weak $\lim_{t \to 0+} (x - J_t x)/t$ and the weak $\lim_{t \to 0+} (x - S(t)x)/t$ exist and are equal for each x in $D(A)$.

(b) If E^* is (F), then the strong $\lim_{t \to 0+} (x - J_t x)/t$ and the strong $\lim_{t \to 0+} (x - S(t)x)/t$ exist and are equal.

Proof. Let x belong to $D(A)$, and let j_t belong to $J((x - J_t x)/t)$. Recall that $p(x) = \lim_{t \to 0+} |x - J_t x|/t = \lim_{t \to 0+} |x - S(t)x|/t \le \|Ax\|$ exists. Suppose that subnets of $\{(x - J_s x)/s\}$ and $\{(x - S(s)x)/s\}$ converge weakly as $s \to 0+$ to z and y respectively. Then $|z| \le p(x)$ and $|y| \le p(x)$. On the other hand, $(z, j_t) \ge |x - J_t x|^2/t^2$ and $(y, j_t) \ge |x - J_t x|^2/t^2$ by (2.2) and (3.3). It follows that, in fact, $|z| = |y| = p(x)$. Now let a subnet of $\{j_t\}$ converge weakly to j as $t \to 0+$. Then $|j| \le p(x)$ and $(z, j) \ge [p(x)]^2$. Hence $|j| = p(x)$ and $(z, j) = (y, j) = [p(x)]^2$. In other words, both z and y belong to $J_{E^*}(j)$. Since E is reflexive and strictly convex, E^* is smooth, $J_{E^*}(j)$ is a singleton and (a) follows. Part (b) follows from (a) and Lemma 1.1.

Theorem 6.1 improves upon a recent result of Plant [12] who proved Part (b) for uniformly convex E by a different argument. It remains valid if $D(A)$ is replaced by the generalized domain $\hat{D}(A)$ of Crandall [5].

In the setting of Part (b) of Theorem 6.1, assume in addition that A is closed and that E is smooth. Let $A^0 x = \{y \in Ax : |y| = \|Ax\|\}$ be the canonical restriction of A. Since $\lim_{t \to 0+} J_t x = x$ for all x in $c\ell(D(A))$, we see that $\hat{D}(A) = D(A)$, the common limit of Part (b) belongs to $A^0 x$, and $p(x) = \|Ax\|$. Let B be a maximal accretive extension of A in $c\ell(D(A))$. Since $p(x)$ is not changed for $x \in D(A)$, $\|Bx\| = \|Ax\|$.

But B^0x is a singleton because Bx is closed and convex. Therefore A^0x is a singleton too and $\lim_{t\to 0}(x - J_t x)/t = \lim_{t\to 0+}(x - S(t)x)/t = A^0x$ for all x in $D(A)$.

Corollary 6.2. Let E be a Banach space, $A \subset E \times E$ an accretive operator that satisfies the range condition, and S the semigroup generated by $-A$. If E^* is (F), E is smooth, and A is closed, then the (negative) infinitesimal generator of S is equal to the canonical restriction of A.

This result is of interest in connection with our Hille-Yosida theorem for semigroups on arbitrary closed convex subsets of E [23].

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Let E be a Banach space, $A \subset E \times E$ an accretive operator that satisfies the range condition, J_r the resolvent of A, and S the nonlinear semigroup generated by $-A$. It is shown that if the norm of E^* is Fréchet differentiable, then $\lim_{t\to\infty} J_t x/t = \lim_{t\to\infty} S(t)x/t$ for each x in $c\ell(D(A))$, and
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ABSTRACT (continued)

$\lim_{t\to 0+} (x - J_t x)/t = \lim_{t\to 0+} (x - S(t)x)/t$ for each $x \in D(A)$. All limits are taken in the norm topology. If E is also smooth, then the first common limit is $-v$, where v is the unique point of least norm in $c\ell(R(A))$. If, in addition, A is closed, then the second common limit is $A^0 x$, the unique point of least norm in Ax. We also show that if A is m-accretive and E^* is strictly convex, then $c\ell(R(A))$ is convex.